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PAIRWISE ORTHOGONAL F-RECTANGLE DESIGNS

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Abstract: The concept of pairwise orthogonal Latin square designs is applied to r row by c column experiment designs which are called pairwise orthogonal F-rectangle designs. These designs are useful in designing successive and/or simultaneous experiments on the same set of rc experimental units, in constructing codes, and in constructing orthogonal arrays. A pair of orthogonal F-rectangle designs exists for any set of v treatment (symbols), whereas no pair of orthogonal Latin square designs of order two and six exists; one of the two construction methods presented does not rely on any previous knowledge about the existence of a pair of orthogonal Latin square designs, whereas the second one does. It is shown how to extend the methods to $r = pv$ row by $c = qv$ column designs and how to obtain t pairwise orthogonal F-rectangle designs. When the maximum possible number of pairwise orthogonal F-rectangle designs is attained the set is said to be complete. Complete sets are obtained for all v for which v is a prime power. The construction method makes use of the existence of a complete set of pairwise orthogonal Latin square designs and of an orthogonal array with v^n columns, $(v^n - 1)/(v - 1)$ rows, v symbols, and of strength two.

1. Introduction and Summary

The existence of complete sets of pairwise orthogonal Latin squares of order n, a prime power, has been known for over 60 years; see, e.g., MacNeish (1922). The existence of complete sets of pairwise orthogonal F-squares of order $n = s^m$ with s treatments (symbols) for s a prime power was demonstrated by Hedayat *et al.* (1975), while the existence of complete sets of F-squares of order 4t, t = 1, 2, ..., with two treatments was proved by Federer (1977). Mandeli (1975) showed how to construct complete sets of pairwise orthogonal F-squares with a variable number of treatments for prime powers. Mandeli *et al.* (1981) showed how to construct sets of pairwise orthogonal F-squares of order $n = 2s^m$ with s treatments and for s a prime power. The set was not complete, but became asymptotically complete as s and/or m approached infinity. Cheng (1980) and Mandeli and Federer (1981) presented results on the construction of complete sets of orthogonal F-hyper-rectangle designs for the number of treatments a prime power. The present paper applies and extends the results of the last two papers. Research on orthogonality of F-rectangles appears to be mostly limited to these three papers. In practice, sets of two and of three orthogonal F-rectangles have been designed for marketing studies.

During the conduct of investigations, r row by c column experiment designs with v treatments may be conducted simultaneously and/or sequentially on the same set of experimental units. The question of existence of pairwise orthogonal r row by c column designs with the same v or different v treatments arises. We call a r row by c column design with v treatments a F-rectangle design (FRD). We show how to construct a pair of orthogonal FRDs for any v. Then, we show how to construct t pairwise orthogonal FRDs for any t for which t pairwise orthogonal Latin squares [POLS(v,t)] exist. Also, we show how to construct a complete set of pairwise orthogonal FRDs for v = 2, r = 2, and c = 4k; this set exists for all 4k for which a Hadamard matrix exists. It is further shown how to construct the complete set of pairwise orthogonal FRDs for v a prime power and how to construct the set (not complete) of pairwise orthogonal FRDs for which a POLS(v,r)-set exists. Then it is shown how to decompose a set of pairwise orthogonal FRDs into pairwise orthogonal FRDs with smaller numbers of treatments. Finally, we point out the application of these results to coding theory and to orthogonal arrays. Codes may be constructed which have length v^{n+1} and width v^n as well as of other dimensions. Definitions in the above cited references are used here and are not given to save space.

2. Pair of orthogonal F-rectangle designs for any v

It is well known that at least a pair of orthogonal Latin squares exists for all Latin squares of order v except v = 2, 6. The question arises concerning the existence of a pair of orthogonal F-rectangles for v treatments (symbols). The question can be answered in the affirmative for any v, even 2 and 6, as indicated in the following theorem.

Theorem 2.1. *For every v, there exists a pair of orthogonal v × 2v F-rectangle designs.*

Proof. For every v except 2 and 6, there exists a pair of orthogonal Latin squares of order v. Denote these as $L_1(v) = L_1$ and $L_2(v) = L_2$. Then, form two F-rectangles as $F_1 = \boxed{L_1 \mid L_1}$ and $F_2 = \boxed{L_2 \mid L_2}$, or alternatively as $\boxed{L_1 \mid L_2}$ and $\boxed{L_2 \mid L_1}$. Obviously, F_1 and F_2 are orthogonal to each other from the property of pairwise orthogonal Latin squares. For v = 2, we exhibit a pair of 2 × 4 orthogonal F-rectangles:

$$F_1 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} .$$

Now for v = 6 construct F_1 by placing side by side two cyclic Latin squares of order 6 in standard form as follows:

$$F_1 = \left|\begin{array}{cccccccccccc} 1&2&3&4&5&6&1&2&3&4&5&6\\ 2&3&4&5&6&1&2&3&4&5&6&1\\ 3&4&5&6&1&2&3&4&5&6&1&2\\ 4&5&6&1&2&3&4&5&6&1&2&3\\ 5&6&1&2&3&4&5&6&1&2&3&4\\ 6&1&2&3&4&5&6&1&2&3&4&5 \end{array}\right| = \boxed{L_1(6)}\boxed{L_1(6)} .$$

Now write out a cyclic Latin square of order 6 with ones on the main right diagonal and write out a second cyclic Latin square of order 6 with twos on the main right diagonal. Place these two Latin squares of order 6 side by side as follows:

$$F_2 = \left|\begin{array}{cccccccccccc} 1&2&3&4&5&6&2&3&4&5&6&1\\ 6&1&2&3&4&5&1&2&3&4&5&6\\ 5&6&1&2&3&4&6&1&2&3&4&5\\ 4&5&6&1&2&3&5&6&1&2&3&4\\ 3&4&5&6&1&2&4&5&6&1&2&3\\ 2&3&4&5&6&1&3&4&5&6&1&2 \end{array}\right| = \boxed{L_2(6)}\boxed{L_2(6)} .$$

Now, F_2 is $\perp$ to F_1. The above procedure may be used for any v except v = 2. This is interesting because a pair of orthogonal F-rectangle designs of v rows by 2v columns may be constructed without relying on the knowledge that a pair of orthogonal Latin squares exists.

It should be noted that the pair of orthogonal 6 X 12 F-rectangles is not unique. Below is a pair that is nonisomorphic to the above pair:

1	2	3	4	5	6	1	2	3	4	5	6	1	3	5	2	4	3	5	1	6	4	6	2
6	1	2	3	4	5	6	1	2	3	4	5	4	2	4	6	3	5	3	6	2	1	5	1
5	6	1	2	3	4	5	6	1	2	3	4	6	5	3	5	1	4	2	4	1	3	2	6
4	5	6	1	2	3	4	5	6	1	2	3	5	1	6	4	6	2	1	3	5	2	4	3
3	4	5	6	1	2	3	4	5	6	1	2	3	6	2	1	5	1	4	2	4	6	3	5
2	3	4	5	6	1	2	3	4	5	6	1	2	4	1	3	2	6	6	5	3	5	1	4

Several more pairs can be formed by taking $L_1(6)$ as one of the 17 squares given in Fisher and Yates (1938). The second pair of orthogonal 6×12 F-rectangles was obtained by trial and error, whereas a procedure is given for obtaining the first pair.

Theorem 2.1 can be generalized as follows:

Theorem 2.2. There exists a pair of orthogonal r-row by c-column F-rectangle designs for

(i) *any* v *when* r *is a multiple of* v *and* c *is a multiple of* 2v

and

(ii) *any* $v \neq 2, 6$, *when* r *and* c *are multiples of* v.

Proof. For any v, construct a pair of $v \times 2v$ F-rectangles as above and denote these as F_1 and F_2. Then for r, a multiple of v and c, a multiple of 2v, construct F_1^* and F_2^* as

$$F_1^* = \begin{bmatrix} F_1 & F_1 & \cdots \\ F_1 & F_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad F_2^* = \begin{bmatrix} F_2 & F_2 & \cdots \\ F_2 & F_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} .$$

Since $F_1 \perp F_2$, then $F_1^* \perp F_2^*$.

For $v \neq 2, 6$, construct $F_1^* \perp F_2^*$ as follows:

$$F_1^* = \boxed{\begin{array}{ccc} L_1 & L_1 & \cdots \\ L_1 & L_1 & \cdots \\ \vdots & \vdots & \ddots \end{array}} \quad \text{and} \quad F_2^* = \boxed{\begin{array}{ccc} L_2 & L_2 & \cdots \\ L_2 & L_2 & \cdots \\ \vdots & \vdots & \ddots \end{array}} .$$

Since $L_1 \perp L_2$, then $F_1^* \perp F_2^*$.

Before proceeding to construct additional F-rectangles which are pairwise orthogonal some notation is required. A well established notation for t pairwise orthogonal Latin squares of order v is POLS(v,t). For t pairwise orthogonal F-square designs of order n, we use the notation POFSD$(n;\lambda_1,\lambda_2,\cdots,\lambda_v;t)$ where λ_i is the frequency with which the i'th treatment (symbol) occurs in each row and each column, $i = 1, 2, \cdots, v$ = number of treatments. For F-rectangles with r rows and c columns, it will be necessary to indicate the values of r and c as well as the frequency of occurrence in rows π_i and the frequency of occurrence in columns λ_i. For t pairwise orthogonal F-rectangles we use the notation POFRD$(r,c;\pi_1,\cdots,\pi_v,\lambda_1,\cdots,\lambda_v;t)$. When $r = v$, this may be simplified to POFRD$(c;\lambda_1,\cdots,\lambda_v;t)$ and when the λ_i are also equal, we use POFRD$(c;\lambda^v;t)$. For the last situation a simple change-over design (SCOD) results. (See, e.g., Federer, 1955, and Kershner and Federer, 1981.)

3. A set of t pairwise orthogonal F-rectangle designs

Given that a POLS(v,t)-set exists, one can write the following theorem.

Theorem 3.1. *A set of t pairwise orthogonal pv by qv F-rectangles exists for every POLS(v,t) set.*

Proof: Let L_i, $i = 1, 2, \cdots, t$, be the t pairwise orthogonal Latin squares of order v in the set POLS(v,t). Construct F-rectangle F_i as follows:

$$F_i = \begin{bmatrix} L_i & L_i & \cdots & L_i \\ L_i & L_i & \cdots & L_i \\ \vdots & \vdots & \ddots & \vdots \\ L_i & L_i & \cdots & L_i \end{bmatrix} .$$

F_i is pv X qv and is denoted by $FRD_i(pv, qv; q^v, p^v)$, since each treatment occurs q times in each row and p times in each column. The set of t orthogonal F-rectangle designs is denoted as $POFRD(pv, qv; q^v, p^v; t)$. For all $v \neq 2, 6$, $2 \leq t \leq v - 1$. When $p = 1$, a simple change-over design (SCOD) results.

Now the question arises concerning other values as well as the maximal value of t. In this connection we can say the following:

Theorem 3.2. *The maximal value of t is the integer part of* $(r-1)(c-1)/(v-1)$.

Corollary 3.1. *The maximal value of t for* $p = 1$ *and* $c = qv$ *is* $qv - 1$.

Proof. In an $r = pv$ by $c = qv$ row by column design, there are $(pv - 1)(qv - 1)$ degrees of freedom associated with the row by column interaction. Each set of treatments in a FRD is associated with $v - 1$ degrees of freedom, and each of the t sets of the $v - 1$ degrees of freedom must come from the interaction degrees of freedom in order to be orthogonal to row and column contrasts. Hence, there are at most $(r-1)(c-1)/(v-1) = (pv-1)(qv-1)/(v-1)$ sets. When $p = 1$, the maximal value for t is $qv - 1$; note that these are the simple change-over designs (SCODs).

Definition 3.1. When $t = (pv-1)(qv-1)/(v-1)$, the set $POFRD(pv, qv; q^v, p^v; t)$ is said to be complete.

4. Complete sets of pairwise orthogonal F-rectangles for $v = 2$, $p = 1$

In a simple change-over design with $v = 2$ symbols, there are two rows and 2q columns. Now, when $2q = 4k$, $k = 1, 2, \cdots$, a complete set of pairwise mutually orthogonal FRDs exists as described below.

Theorem 4.1. A POFRD($4k; (2k)^2; 4k-1$) set exists for all $4k$ for which a Hadamard matrix exists.

Proof. In a FRD($4k; (2k)^2$), there are two sequences of symbols, namely $\frac{1}{2}$ and $\frac{2}{1}$ in the $4k$ columns. Denote one of the sequences as +1 and the other as -1. When a Hadamard matrix is normalized there are $4k$ plus ones in the first column and in the first row. In the second through the $4k$'th row, there are $2k$ plus ones and $2k$ minus ones, and every row is orthogonal to every other row. Now construct $4k - 1$ FRDs from the last $4k - 1$ rows of the Hadamard matrix where a plus one indicates the sequence $\frac{1}{2}$ and a minus one indicates the sequence $\frac{2}{1}$. Since any two rows of the Hadamard matrix are orthogonal, any two corresponding two FRDs will be orthogonal. Since $4k - 1 = t$ is the maximum number of FRDs that can be constructed, the set is complete. Hence, a POFRD($4k; (2k)^2; 4k-1$) set exists if a Hadamard matrix of order $4k$ exists.

Now we can also prove the following.

Theorem 4.2. $t = 0$ or 1 for all $2q \neq 4k$, $k = 1, 2, \cdots$.

Proof. When the number of columns is equal to $4k - 1$ or $4k - 3$, $k = 1, 2, \cdots$, no FRD exists, i.e., $t = 0$. When $2q = 4k - 2$, $k = 1, 2, \cdots$, one can easily construct a FRD; hence, t is at least one. Now, in constructing +1 and -1 $(4k - 2) \times (4k - 2)$ contrast matrices containing $(2k - 1)$ plus ones and $(2k - 1)$ minus ones, one may construct the first row with all plus ones and the second row with $(2k - 1)$ plus ones and $(2k - 1)$ minus ones. Now it is impossible to construct a third row of the matrix which has $(2k - 1)$ plus ones and $(2k - 1)$ minus ones and which is orthogonal to each of the first two rows of the matrix. This is so because it is impossible to divide an odd number, $2k - 1$, into two equal parts. Since this is not possible, $t = 1$ for all $4k - 2$. Note that when $k = 1$, we have a 2×2 Latin square, and we know that it is mateless, i.e., $t = 1$.

5. Complete sets of pairwise orthogonal FRDs for v a prime power, r = v

Prior to presenting the general result for complete sets of pairwise orthogonal F-rectangle designs with v symbols, v a prime power, v rows, and $v^n = qv$ columns, let us consider a POFRD$(9;3^3;8)$-set. To construct this set we use the POLS$(3,2)$-set and the orthogonal array OA$(9,4,3,2)$-set which are:

POLS(3,2)-set		OA(9,4,3,2)		
L_1	L_2	000	111	222
012	012	012	012	012
120	201	012	120	201
201	120	012	201	120

Now use L_1 and associate the symbols 0, 1, 2 in the OA with the columns of L_1. Using the four rows of the OA, we obtain the following four FRDs:

L_1

Row 1 of OA			Row 2 of OA			Row 3 of OA			Row 4 of OA		
000	111	222	012	012	012	012	120	201	012	201	120
111	222	000	120	120	120	120	201	012	120	012	201
222	000	111	201	201	201	201	012	120	201	120	012

Now use L_2 in the same manner to obtain four more FRDs:

L_2

Row 1 of OA			Row 2 of OA			Row 3 of OA			Row 4 of OA		
000	111	222	012	012	012	012	120	201	012	201	120
222	000	111	201	201	201	201	012	120	201	120	012
111	222	000	120	120	120	120	201	012	120	012	201

We now have $qv - 1 = 8$ pairwise orthogonal FRDs, and the set is complete.

Now consider a POFRD$(27;9^3;26)$-set. To construct this set use L_1 and L_2 above and the OA$(27,13,3,2)$ which is

000	000	000	111	111	111	222	222	222
000	111	222	000	111	222	000	111	222
000	111	222	111	222	000	222	000	111
000	222	111	111	000	222	222	111	000
012	012	012	012	012	012	012	012	012
012	012	012	120	120	120	201	201	201
021	021	021	102	102	102	201	201	201
012	120	201	012	120	201	012	120	201
021	102	210	021	102	210	021	102	210
012	120	201	120	201	012	201	012	120
021	102	210	102	210	021	210	021	102
012	201	120	120	012	201	201	120	012
021	210	102	102	021	210	210	102	021

Thus, the POLS$(3,2)$-set and the OA$(27,13,3,2)$ may be used to construct the $3^3 - 1 = 26$ POFRDs which is the complete set.

Following the above procedure we state the following theorem:

Theorem 5.1. *A complete set of pairwise orthogonal F-rectangle designs exists for v a prime power and qv equal to v^n, that is a POFRD$(v^n;(v^{n-1})^v;v^n-1)$-set exists.*

Proof. The proof follows the construction method outlined above. Use a POLS$(v,v-1)$-set and the orthogonal array OA$(v^n,(v^n-1)/(v-1),v,2)$. Take the first Latin square, L_1, from the POLS$(v,v-1)$-set and the first row of OA$(v^n,(v^n-1)/(v-1),v,2)$ to form the first FRD$(v^n;(v^{n-1})^v)$. Take L_1 and the second row of OA to form a second FRD$(v^n;(v^{n-1})^v)$. Continue using rows of OA until $(v^n-1)/(v-1)$ FRD$(v^n;(v^{n-1})^v)$s have been formed. These $(v^n-1)/(v-1)$ FRDs are pairwise orthogonal since the rows of the OA are orthogonal. Now take a second Latin square from the POLS$(v,v-1)$-set and form an additional set of $(v^n-1)/(v-1)$ FRDs. This set forms a pairwise orthogonal set, and is pairwise orthogonal to the first set of $(v^n-1)/(v-1)$ FRDs. Continue this process until the last Latin square in the POLS$(v,v-1)$-set has been used. There will be $(v-1)(v^n-1)/(v-1) = v^n-1$ POFRDs. Since v^n-1 is the maximum number, the set is complete. [Also, see Cheng (1980) and Mandeli and Federer (1981).]

6. Other sets of POFRDs

It is not known what values of t are possible when v is not a prime power and/or $qv \neq v^n$. For example, consider the following three row by six column FRDs:

FRD_1			FRD_2		
00	11	22	00	11	22
22	00	11	11	22	00
11	22	00	22	00	11

It is not known if t can be greater than two in a POFRD$(6;2^3;t)$-set.

For any v, we can state the following:

Theorem 6.1. Given a POLS(v,r)-set and an OA$(v^n,t,v,2)$, the method of construction for Theorem 5.1 produces rt pairwise orthogonal F-rectangle designs, i.e., the POFRD$(v^n;(v^{n-1})^v;rt)$-set.

However, it is not known if the set can be extended for values greater than rt.

7. Decomposition of FRDs

When $v = p^h$, p a prime power and h a positive integer, a v row by v^n column FRD can be decomposed into $(p^h-1)/(p-1)$ POFRDs with p symbols. If an integer k divides h, then the above FRD can be decomposed into $(p^n-1)/(p^k-1)$ POFRDs with p^k symbols. Likewise, for a set of t POFRDs with p^h symbols, each of the t FRDs can be decomposed into $(p^h-1)/(p^k-1)$ sets of POFRDs, resulting in a total of t $(p^h-1)/(p^k-1)$ POFRDs with p^k symbols. One can also decompose this set of t

POFRDs with p^h symbols into sets with variable number of symbols. For example, if $h = 6$, $k = 1, 2, 3$ and 6, resulting in POFRDs with p^6, p^3, p^2, and p symbols. Theorems 7.1 and 7.2 embody the results described above.

Theorem 7.1. If $v = p^h$, where p is a prime power and h is a positive integer, for all integers k which divide h, then a v row by v^n column FRD with v symbols can be decomposed into $(v-1)/(p^k-1)$ POFRDs which are of size v rows by v^n columns and contain p^k symbols.

Theorem 7.2. Given the conditions in Theorem 7.1 for each POFRD_i, $i = 1, 2, \cdots, t$, POFRD_i with p^h symbols can be decomposed into $(p^h-1)/(p^{k_i}-1)$ POFRDs, which are of size v rows and v^n columns and contain p^{k_i} symbols. The t POFRD_is can be decomposed into $\Sigma_{i=1}^t (p^h-1)/(p^{k_i}-1)$ POFRDs of size v rows by v^n columns and variable numbers of symbols p^{k_i}.

The above theorems and their proofs follow from results obtained by Mandeli (1975) and Mandeli and Federer (1981).

Also, partial OAs can be formed from POLS$(v, t < v = 1)$-sets, and they can also be formed from a set of t POFRDs with a variable number of symbols to give an $OA(v^{n+1}, b_1, s_1, 2) + OA(v^{n+1}, b_2, s_2, 2) + \cdots + OA(v^{n+1}, b_a, s_a, 2)$ set.

8. Formation of orthogonal arrays and codes

Just as POLS$(v,v-1)$-sets may be used to construct orthogonal arrays, the POFRD$(v^n; (v^{n-1})^v; v^n-1)$-set may also be used to construct arrays of the $OA(v^{n+1}, v^n, v, 2) + OA(v^{n+1}, 1, v^n, 2)$ type. Perhaps a better notation for orthogonal arrays with a sets of symbols, $s_1, s_2, \cdots, s_a$, $b_1, b_2, \cdots, b_a$ rows (assemblies) with s_i symbols being associated with b_i rows, and cr runs, would be $OA(cr; b_1, b_2, \cdots, b_a; s_1, s_2, \cdots, s_a; 2)$. For example, the orthogonal array formed from the pair of orthogonal 6×12 rectangles would be $OA(72; 1, 3; 12, 6; 2)$. That is, there would be one row with 12 symbols and 3 rows with 6 symbols. These orthogonal arrays are then used to construct codes in the same manner as they are for the OAs formed from POLS(v,t)-sets.

The set of POFRDs obtained from Theorems 7.1 and 7.2 can be used to construct orthogonal arrays with p^{k_i} symbols for all k_i which divide h. Likewise, codes from these orthogonal arrays can be constructed with variable numbers of symbols.

A previous limitation in constructing codes was the width of the orthogonal array. This limitation has now been removed in that the width of the code for v symbols, v a prime power, is v^n where n may be any positive integer. The length of the code has been no problem, since the orthogonal array may be repeated as often as required. Also, the above results allow construction of codes with variable numbers of symbols. This problem is treated in detail by Federer and Mandeli (1983).

Remark

The above discussion was confined in some instances to FRDs which had v rows. The results can easily be extended to the case where there are v^m rows and v^n columns in the FRDs [see, e.g., Mandeli and Federer (1981)].

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